ON THE THEORY OF ROTATING MAGNETIC STARS. PART I.

The kinetical equations of stars considered as conducting gases may be written under very general conditions. These equations not only permit the consideration of axial rotation and of meridional currents but they also offer an utterly general structure for the magnetic field. Together with the component of the magnetic field parallel to the meridional plane (mentioned in the following as "meridional component"), the component perpendicular to the meridional plane (φ-component) may be introduced as well. The introduction of the latter may be only necessitated by the general hydro- and electrodynamical conditions.

1. *Basic equations*

The equations of magneto-hydrodynamics serve as point of departure. In a stationary state these are the following :

$$(\mathfrak{v}, \text{grad})\, \mathfrak{v} - (\mathfrak{H}, \text{grad})\, \mathfrak{H} = \text{grad}\, V - \frac{1}{\varrho}\, \text{grad}\, P + \nu \Delta\, \mathfrak{v} \tag{1}$$

$$\text{rot}\, [\mathfrak{v}, \mathfrak{H}] = \varkappa \Delta\, \mathfrak{H} \tag{2}$$

$$\text{div}\, \mathfrak{H} = 0 \tag{3}$$

$$\text{div}\, \varrho\, \mathfrak{v} = 0\ . \tag{4}$$

Supposing that the velocity, the magnetic field and the other quantities depend only upon the coordinates in the meridional plane r and ϑ, and are independent of the position of the meridional plane (coordinate φ), equation (2) may be written thus :

$$\text{rot}\, [\mathfrak{v}_m, \mathfrak{H}_m] = \varkappa \Delta\, \mathfrak{H}_m \tag{5}$$

and

$$R\left(\mathfrak{v}_m, \text{grad}\, \frac{\mathfrak{H}_\varphi}{R}\right) - R\left(\mathfrak{H}_m, \text{grad}\, \frac{\mathfrak{v}_\varphi}{R}\right) = \varkappa\left(\Delta\, \mathfrak{H}_\varphi - \frac{\mathfrak{H}_\varphi}{R^2}\right), \tag{6}$$

where $\mathfrak{v}_m$ and $\mathfrak{H}_m$ are the meridional components of the velocity and of the magnetic field, whereas $\mathfrak{v}_\varphi$ and $\mathfrak{H}_\varphi$ are the components perpendicular to the meridional plane.

Supposing that $\mathfrak{H}_\varphi = 0$, the latter equation takes a very simple form :

$$(\mathfrak{H}_m, \text{grad}\, \omega) = 0, \tag{7}$$

($\omega = \mathfrak{v}_\varphi / R$ meaning angular velocity.) This corresponds to Alfvén's result which states the coincidence of the lines of force (i. e. surfaces of force) of the magnetic field with the surfaces of constant angular velocity.

Let us express the meridional component of the magnetic field by vector potential

$$\mathfrak{H}_m = \text{rot}\, \mathfrak{A}_\varphi, \tag{8}$$

where $\mathfrak{A}_\varphi$ is a vector perpendicular to the meridional plane. Placing (8) into equation (7) which we write in cylindrical coordinates, we obtain :

$$\frac{\partial \mathfrak{A}_\varphi}{\partial z} \frac{\partial \omega}{\partial R} - \frac{1}{R} \frac{\partial}{\partial R} (R\, \mathfrak{A}_\varphi) \frac{\partial \omega}{\partial z} = 0.$$

The left side is the Jacobian of the functions ω and $R\, \mathfrak{A}_\varphi$, which, being equal to zero, the two functions are related in the following manner

$$\omega = F\, (R\, \mathfrak{A}_\varphi) \tag{9}$$

where F is an arbitrary function.

But in a more general system the supposition $\mathfrak{H}_\varphi = 0$ is not valid, consequently Alfvén's theory as well as relation (9) cannot stand either.

Let us simplify equation (5) in the following manner

$$\text{rot}\, \big([\mathfrak{v}_m, \mathfrak{H}_m] - \varkappa\, \text{rot}\, \mathfrak{H}_m \big) = 0$$

which means that the bracketed formula may also be expressed by a gradient function

$$[\mathfrak{v}_m, \mathfrak{H}_m] - \varkappa\, \text{rot}\, \mathfrak{H}_m = \text{grad}\, \Phi\ .$$

As the right side of the equation means a vector perpendicular to the meridional plane and as the component of the left side in this direction equals identically zero, we may write

$$[\mathfrak{v}_m, \mathfrak{H}_m]_\varphi - \varkappa\, (\text{rot}\, \mathfrak{H}_m)_\varphi = 0,$$

which is identical to the φ-component of the first Maxwellian equation. By introducing the vector potential $\mathfrak{A}_\varphi$ according to (8), we obtain

$$[\mathfrak{v}_m, \text{rot}\, \mathfrak{A}_\varphi]_\varphi = -\, \varkappa \left(\Delta\, \mathfrak{A}_\varphi - \frac{\mathfrak{A}_\varphi}{R^2} \right).$$

Or, introducing cylindrical coordinates, we get

$$\mathfrak{v}_z\, \frac{\partial \mathfrak{A}_\varphi}{\partial z} + \mathfrak{v}_R \frac{1}{R} \frac{\partial}{\partial R} (R\, \mathfrak{A}_\varphi) = -\, \varkappa \left(\Delta\, \mathfrak{A}_\varphi - \frac{\mathfrak{A}_\varphi}{R^2} \right). \tag{10}$$

Now, for instance, if conductivity is infinite large i. e. $\varkappa = 0$, the left side of (10) equals zero. Introducing the vector potential of the velocity of the meridional currents, by the equation

$$\mathfrak{v}_m = \text{rot}\, \mathfrak{B}_\varphi, \tag{11}$$

the left side of (10) will be

$$\frac{1}{R} \frac{\partial}{\partial R} (R\, \mathfrak{B}_\varphi) \frac{\partial \mathfrak{A}_\varphi}{\partial z} - \frac{\partial \mathfrak{B}_\varphi}{\partial z} \frac{1}{R} \frac{\partial}{\partial R} (R\, \mathfrak{A}_\varphi) = 0,$$

from which it follows that

$$R\,\mathfrak{A}_\varphi = G\,(R\,\mathfrak{B}_\varphi),$$

and, comparing the above with (9) we get

$$\omega = H\,(R\,\mathfrak{B}_\varphi),$$

which means that in this case the angular velocity may be considered as a function of $R\,\mathfrak{B}_\varphi$ only.

Lastly the φ-component of equation (1)

$$\mathfrak{v}_R \frac{1}{R}\frac{\partial}{\partial R}(R\,\mathfrak{v}_\varphi) + \mathfrak{v}_z \frac{\partial\,\mathfrak{v}_\varphi}{\partial z} - \mathfrak{H}_R \frac{1}{R}\frac{\partial}{\partial R}(R\,\mathfrak{H}_\varphi) - \mathfrak{H}_z \frac{\partial\,\mathfrak{H}_\varphi}{\partial Z} =$$

$$= \varkappa\left(\Delta\,\mathfrak{v}_\varphi - \frac{\mathfrak{v}_\varphi}{R^2}\right) \qquad (12)$$

in the case $\mathfrak{H}_\varphi = 0$, is also made simpler in its turn and much the same in form as (10)

$$\mathfrak{v}_R \frac{1}{R}\frac{\partial}{\partial R}(R\,\mathfrak{v}_\varphi) + \mathfrak{v}_z \frac{\partial\,\mathfrak{v}_\varphi}{\partial z} = \varkappa\left(\Delta\,\mathfrak{v}_\varphi - \frac{\mathfrak{v}_\varphi}{R^2}\right). \qquad (13)$$

Thus in this special case our task is the following: solutions must be found for equation (10) and (13) satisfying the condition

$$\frac{\mathfrak{v}_\varphi}{R} = F\,(R\,\mathfrak{A}_\varphi),$$

through a vector potential $\mathfrak{B}_\varphi$ equalling zero at the boundaries of the meridian quadrant.

Supposing further that the viscosity coefficient equals zero, the equation

$$\frac{\partial\,\mathfrak{B}_\varphi}{\partial z}\frac{1}{R}\frac{\partial}{\partial R}(R\,\mathfrak{v}_\varphi) - \frac{1}{R}\frac{\partial}{\partial R}(R\,\mathfrak{B}_\varphi)\frac{\partial\,\mathfrak{v}_\varphi}{\partial z} = 0 \qquad (14)$$

gives the relation

$$R\,\mathfrak{v}_\varphi = f\,(R\,\mathfrak{B}_\varphi),$$

which, when compared with the relation received for the case $\varkappa = 0$ gives

$$\frac{\mathfrak{v}_\varphi}{R} = g\,(R\,\mathfrak{v}_\varphi).$$

From this it follows that $\mathfrak{v}_\varphi$, consequently $\mathfrak{A}_\varphi$ and $\mathfrak{B}_\varphi$ as well, depend only upon the distance from the axis.

This paper deals with the simplified case where the component of the magnetic field perpendicular to the meridional plane vanishes identically. Thus the basic equations are the following:

$$-\frac{\partial\,\mathfrak{B}_\varphi}{\partial z}\frac{1}{R}\frac{\partial}{\partial R}(R\,\mathfrak{v}_\varphi) + \frac{1}{R}\frac{\partial}{\partial R}(R\,\mathfrak{B}_\varphi)\frac{\partial\,\mathfrak{v}_\varphi}{\partial z} = \nu\left(\Delta\,\mathfrak{v}_\varphi - \frac{\mathfrak{v}_\varphi}{R^2}\right) \qquad (15)$$

$$-\frac{\partial \mathfrak{B}_\varphi}{\partial z}\frac{1}{R}\frac{\partial}{\partial R}(R\mathfrak{A}_\varphi)+\frac{1}{R}\frac{\partial}{\partial R}(R\mathfrak{B}_\varphi)\frac{\partial \mathfrak{A}_\varphi}{\partial z}=\varkappa\left(\Delta\mathfrak{A}_\varphi-\frac{\mathfrak{A}_\varphi}{R^2}\right) \tag{16}$$

$$\frac{\mathfrak{v}_\varphi}{R}=F(R\mathfrak{A}_\varphi) \tag{17}$$

$\mathfrak{v}_\varphi$ and $\mathfrak{A}_\varphi$ vanish along the axis of rotation, and further, $\mathfrak{B}_\varphi$, the vector potential of the velocity of the meridional currents vanishes all along the boundaries of the meridional quadrant.

2. *Solution of the model* $\mathfrak{H}_\varphi = 0$

In the case $\nu = \varkappa$, one of the solutions of the initial equations (15), (16) (17), satisfying the boundary conditions, is

$$\mathfrak{v}_\varphi = C\,\mathfrak{A}_\varphi, \tag{18}$$

which, when placed into equations (17) gives the following functional equation

$$\frac{\mathfrak{v}_\varphi}{R} = F\,(C\,R\,\mathfrak{v}_\varphi), \tag{19}$$

the solution of which is

$$\mathfrak{v}_\varphi = C\,\Phi\,(R). \tag{20}$$

(According to (18) however $\mathfrak{A}_\varphi = \Phi\,(R)$. In course of our further investigation of the case $\nu = \varkappa$, we shall write down only relations for $\mathfrak{v}_\varphi$). Considering (20) equation (15) may thus be simplified

$$-\frac{\partial \mathfrak{B}_\varphi}{\partial z}\frac{1}{R}\frac{d}{dR}(R\,\mathfrak{v}_\varphi)=\nu\left(\frac{1}{R}\frac{d}{dR}\left(R\frac{d\,\mathfrak{v}_\varphi}{d\,R}\right)-\frac{\mathfrak{v}_\varphi}{R^2}\right). \tag{21}$$

From this it follows

$$\frac{\partial \mathfrak{B}_\varphi}{\partial z} = -\frac{\nu}{R}\,b\,(R),$$

i. e.

$$\mathfrak{B}_\varphi = -\frac{\nu}{R}\,b\,(R)\,z. \tag{22}$$

Applying the substitution $\frac{d\,(R\,\mathfrak{v}_\varphi)}{d\,R} = y$ in (21) and by using (22), we get

$$\frac{b\,(R)+1}{R}\,y = y',$$

the solution of which is

$$y = c\,R\,e^{\int\frac{b\,(R)\,d\,R}{R}}.$$

This means that

$$\mathfrak{v}_\varphi = \frac{c}{R} e^{\int \frac{b(R)\,dR}{R}} dR. \tag{23}$$

The stream lines of the meridional currents are given by equation

$$[\text{rot}\, \mathfrak{B}_\varphi, d\,\mathfrak{r}] = 0,$$

$d\mathfrak{r}$ meaning an element of the stream line. From this it follows that the equation of the stream line is

$$R\, \mathfrak{B}_\varphi = \text{constant},$$

and so

$$z = \frac{C}{\nu\, b\,(R)}. \tag{24}$$

The shape of the stream line cannot be given unless function $b\,(R)$ is known. We may, however, draw some very important conclusions from the above form of (24); if, in case of R being constant, we examine z as a function of C, we shall find that, proceeding parallelly along the Z-axis, C takes all values between 0 and ∞ only once, which means that we have cut every stream line only once. Consequently the stream lines along the meridian quadrants do not form a closed family of curves but a system of lines approaching the axis asymptotically. Meridional currents of this type would lead to the disruption of the star. A velocity-distribution of this type is therefore impossible.

Except for the supposition $\mathfrak{H}_\varphi = 0$, our investigation contained no other essential limitation. Thus we must conclude that the supposition $\mathfrak{H}_\varphi = 0$ is not tenable in the case of a rotating star and so (7) is not to be maintained either, the latter expressing through the introduction of the above supposition the coincidence of the magnetic line of force (surfaces of force) and the surfaces of constant angular velocity.

Should $\mathfrak{H}_\varphi \neq 0$, equation (6) must be used for its determination. This problem shall be dealt with in another paper.

For the case of completeness we shall just give one example for the model $\mathfrak{H}_\varphi = 0$ (in case of $\nu \neq \varkappa$).

3. *Illustrative solution for the case* $\nu \neq \varkappa$

Should F be a linear function of $R\,\mathfrak{v}_\varphi$ in (19), a simple procedure gives the result

$$\mathfrak{v}_\varphi = \frac{\omega_0 R}{r_0^2 + \omega_1 R^2}, \tag{25}$$

and from (22)

$$\mathfrak{B}_\varphi = \frac{4\,\omega_1\, \nu\, R\, z}{r_0^2 + \omega_1 R^2}. \tag{26}$$

But this solution stands only for case $\nu = \varkappa$ because, substituting (25) for (15) and (16) separately, we get the following result for $\mathfrak{B}_\varphi$:

$$\mathfrak{B}_\varphi = \frac{4\,\omega_1\, \nu\, R\, z}{r_0^2 + \omega_1 R^2}$$

$$\mathfrak{B}_\varphi = \frac{4\,\omega_1\,\varkappa\,R\,z}{r_0^2 + \omega_1 R^2}, \tag{27}$$

and these, in case $\nu = \varkappa$, really take the same form. For the case $\nu \neq \varkappa$, we substitute for ν and $\varkappa$ in (27) the function $f(\nu,\varkappa)$ which takes the form of $f = \nu\,(=\varkappa)$ in case of $\nu = \varkappa$. Functions of that type are for instance

$$f = \frac{1}{2}(\nu + \varkappa), \quad f = \sqrt{\nu\,\varkappa}, \quad f = \frac{2\,\nu\,\varkappa}{\nu + \varkappa} \text{ etc.}$$

Accordingly, a more general form of the vector potential of the velocity of the meridional currents for the case $\nu \neq \varkappa$ is the following

$$\mathfrak{B}_\varphi = \frac{4\,\omega_1\,f\,(\nu,\varkappa)\,R\,z}{r_0^2 + \omega_1 R_2}.$$

If here too $\mathfrak{v}_\varphi$ and $\mathfrak{A}_\varphi$ are only dependent of R, then, by the substitutions $\frac{d}{d\,R}(R\,\mathfrak{v}_\varphi) = x$, and $\frac{d}{d\,R}(R\,\mathfrak{A}_\varphi) = y$, we get the two differential equations:

$$x' = x\,\frac{r_0^2 - \omega_1\left[4\,\frac{f\,(\nu,\varkappa)}{\nu} - 1\right]R^2}{(r_0^2 + \omega_1 R^2)\,R}$$

$$y' = y\,\frac{r_0 - \omega_1\left[4\,\frac{f\,(\nu,\varkappa)}{\nu} - 1\right]R^2}{(r_0^2 + \omega_1 R^2)\,R},$$

the solutions of which are

$$x = \frac{c_1 R}{(r_0^2 + \omega_1 R^2)^{\frac{2f}{\nu}}}.$$

$$y = \frac{c_1 R}{(r_0^2 + \omega_1 R^2)^{\frac{2f}{\varkappa}}}.$$

Solutions free of singularity may not be obtained for arbitrary values of ν and $\varkappa$ unless

$$f = \frac{2\,\nu\,\varkappa}{\nu + \varkappa}$$

(i. e. an odd function of $\nu\varkappa$), in which case

$$\mathfrak{v}\,\varphi = -\frac{(\nu + \varkappa)\,\omega_0}{(3\,\varkappa - \nu)\,\omega_1}\left\{\frac{1}{R}\,\frac{1}{(r_0^2 + \omega_1 R^2)^{\frac{3\varkappa - \nu}{\nu + \varkappa}}} - \frac{1}{R\,(r_0^2)^{\frac{3\varkappa - \nu}{\nu + \varkappa}}}\right\} \tag{28}$$

$$\mathfrak{A}\,\varphi = -\frac{(\nu + \varkappa)\,\omega_0}{(3\,\varkappa - \nu)\,\omega_1}\left\{\frac{1}{R}\,\frac{1}{(r_0^2 + \omega_1 R^2)^{\frac{3\nu - \varkappa}{\nu + \varkappa}}} - \frac{1}{R\,(r_0^2)^{\frac{3\nu - \varkappa}{\nu + \varkappa}}}\right\} \tag{29}$$

In the special case $\nu = \varkappa$, accordingly to (18), we get a similar expression both for $\mathfrak{v}_\varphi$ and $\mathfrak{B}_\varphi$. In this case both $\mathfrak{B}_\varphi$ and $\mathfrak{H}_m$ ($=$ rot $\mathfrak{A}_\varphi$) decrease at an equal rate with the extent of their distance from the axis. But if $\nu > \varkappa$, the decrease of velocity will be smaller whereas the decrease of the vector potential of the magnetic field will be larger. The change will be the reverse in case $\nu > \varkappa$. In the special case $\nu = 3\varkappa$, $\mathfrak{v}_\varphi$ will correspond to uniform rotation while the vector potential of the magnetic field takes the form

$$\mathfrak{A}_\varphi = \frac{1}{2}\omega_0 R \left\{ \frac{1}{r_0^2 + \omega_1 R^2} + \frac{R}{(r_0^2 + \omega_1 R^2)^2} \right\}.$$

In the case $\varkappa = 3\nu$, the magnetic field will be uniform whereas the velocity diminishes rapidly with the distance from the axis.

The relation (18) gives a connection between velocity and magnetic field. If we take the rotation (curl) of both sides, the result will be the more familiar form of this relation :

$$\text{rot}\, \mathfrak{v}_\varphi = C\, \mathfrak{H}_m.$$

But this kind of relation can only be stated in the case of $\nu = \varkappa$. From the solution (28) and (29) containing the more general condition $\nu \neq \varkappa$, no such relations can be deduced.

4. *Conclusions*

1. The vector potential of the velocity of the meridional currents and of the magnetic field depend on $\nu\varkappa$, consequently both quantities will be identically zero, either $\nu = 0$, or $\varkappa = 0$.

2. No model can be found for $\mathfrak{H}_\varphi = 0$, in which case the boundary conditions for the meridional currents cannot be fulfilled.

3. In the most general model both the distribution of angular velocity and the magnetic field depend also on ν and $\varkappa$. Thus ν and $\varkappa$ may be determined empirically from the observed velocities and magnetic strength on the surface (e. g. solar surface).

4. According to the theory described in a previous paper * in magneto-hydrodynamical turbulence $\varkappa$ is of the same order as the coefficient of eddy viscosity (i. e. eddy conductivity being very small). Consequently $\varkappa$ exceeds at least 10^{10} times the corresponding molecular value. So the empirical determination of ν and $\varkappa$ leads to a control of the correctness of the theory.

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* Acta Physica Academiae Scientiarum Hungaricae Tomus I. Fasciculus 3. Pag. 235. — Mitteilungen der Sternwarte der ungarischen Akademie der Wissenschaften, Budapest - Szabadsághegy Nr. 26.