ON THE THEORY OF ROTATING MAGNETIC STARS. PART II.

The basic equations of a rotating magnetic star are given in the preceding paper. In the present one the approximate solutions of these equations are sought for under very general conditions. No idealized structure is assumed for producing either the velocity field or the magnetic field. So, besides axial rotation, meridional motions too are examined, and the vector of the magnetic field strength is supposed not only as locating itself in the plane of the meridian, but also as having a component perpendicular to it. This component forms the so-called toroidal megnetic field, whose mathematical investigation explained in the present paper finds itself among the first articles on such a subject.

Whereas the first four parts of this paper contain the pure mathematical discussion of the problem, from the fifth part on the results of the theory are compared with the different solar and astrophysical observations. The theory gives the laws of the distribution of the angular velocity on the solar surface and those of meridional motion correctly. It follows from the theory that the viscosity of the solar matter must be very great, the electric conductivity very small. These results are in harmony with the proprieties guessed till now of hydromagnetical turbulence.

On the ground of the theory an external magnetic dipole field cannot be deduced for the Sun. In accordance with the theory the external field has an octopole structure. From the results it may be concluded that the velocity distribution of the typical magnetic stars cannot be such as that of the Sun, because it is very improbable that the powerful magnetic fields of the stars would arise from a field having a pole of higher order.

A relation between the radius of the star and its viscosity coefficient can be established. The correctness of this relation is examined by the aid of data of pertinent literature.

1. Introduction

In the preceding paper we have demonstrated that in a magnetic star, besides axial rotation, meridional motions, and the meridional magnetic field, there appears also a field perpendicular to the plane of the meridian. Thus if spatial polar coordinates are introduced, the following components of velocity and magnetic field must be considered:

$$\mathfrak{v}_r,\ \mathfrak{v}_\vartheta,\ \mathfrak{v}_\varphi \text{ — components of } \mathfrak{v},$$

$$\mathfrak{H}_r,\ \mathfrak{H}_\vartheta,\ \mathfrak{H}_\varphi \text{ — components of } \mathfrak{H}.$$

If the system is independent of the coordinate φ, the meridional components of both vectors (r- and ϑ-components) may be expressed by the φ-component of their vector potential:

$$\mathfrak{v}_m = \operatorname{rot} \mathfrak{B}_\varphi,$$
$$\mathfrak{H}_m = \operatorname{rot} \mathfrak{A}_\varphi.$$

Hence the determination of these four functions: $\mathfrak{B}_\varphi$, $\mathfrak{v}_\varphi$, $\mathfrak{A}_\varphi$, $\mathfrak{H}_\varphi$ will be sufficient for the discussion.

The boundary conditions of the velocity field are well-known : $\mathfrak{B}_\varphi$ is zero on the boundary of the meridional quadrant ; $\mathfrak{v}_\varphi$ is zero only along the axis of rotation, in the other points of the space it differs from zero and is symmetric to the plane of the equator.

The initial conditions and distribution of the vector potential $\mathfrak{A}_\varphi$, are similar to those of $\mathfrak{v}_\varphi$. The component $\mathfrak{H}_\varphi$ originates from the electric currents passing on in the meridional plane. These currents start in consequence of the spontaneously arising electric field and the motion (axial rotation) of the conducting gas in the magnetic field. Meridional electric currents, however, form an entirely closed system, for this reason all magnetic lines of force are included in the stellar interior. Hence the boundary conditions of $\mathfrak{H}_\varphi$ are as follows : $\mathfrak{H}_\varphi$ can be continous in the neighbourhood of the surface only if it is zero on the whole surface of the star ; from the circumstance that electric currents are symmetric to the plane of the equator, it follows that $\mathfrak{H}_\varphi$ must be zero also along the axis of rotation and the plane of the equator. Accordingly, $\mathfrak{H}_\varphi$ is zero on the boundary of the meridian quadrant, i. e. its boundary conditions are identical with those of $\mathfrak{B}_\varphi$.

2. *Basic Equations*

In the previous paper the simplification $\mathfrak{H}_\varphi = 0$ was introduced ; for this reason only these three functions : $\mathfrak{B}_\varphi$, $\mathfrak{v}_\varphi$, $\mathfrak{A}_\varphi$ had to be determined. For their discussion three differential equations were deduced. In order to solve also the present problem we start on these three differential equations, which in the more general case ($\mathfrak{H} \neq 0$) are as follows :

a. The first equation is

$$\operatorname{rot} [\operatorname{rot} \mathfrak{B}_\varphi, \mathfrak{H}_\varphi] = \varkappa \Delta \mathfrak{H}, \tag{1}$$

the φ-component of which is

$$-r \sin\vartheta \left\{ \frac{1}{r\sin\vartheta} \frac{\partial}{\partial\vartheta} (\sin\vartheta\, \mathfrak{H}_\varphi) \frac{\partial}{\partial r} \frac{\mathfrak{H}_\varphi}{r\sin\vartheta} + \frac{1}{r}\frac{\partial}{\partial r}(r\mathfrak{B}_\varphi)\frac{1}{r}\frac{\partial}{\partial\vartheta}\frac{\mathfrak{H}_\varphi}{r\sin\vartheta} \right\} +$$

$$+ r\sin\vartheta \left\{ \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\, \mathfrak{A}_\varphi)\frac{\partial}{\partial r}\frac{\mathfrak{v}_\varphi}{r\sin\vartheta} + \frac{1}{r}\frac{\partial}{\partial r}(r\mathfrak{A}_\varphi)\frac{1}{r}\frac{\partial}{\partial\vartheta}\frac{\mathfrak{v}_\varphi}{r\sin\vartheta} \right\} =$$

$$= \varkappa \left(\Delta \mathfrak{H}_\varphi - \frac{\mathfrak{H}_\varphi}{r^2 \sin^2\vartheta} \right). \tag{2}$$

In the previous paper this equation was essentially simpler by reason of the supposition $\mathfrak{H}_\varphi = 0$. There a relation between $\mathfrak{A}_\varphi$ and $\mathfrak{v}_\varphi$ was given by (2), which expressed the coincidence of the magnetic lines of force and the surfaces of constant angular velocity. In the present case (2) serves for the determination of the omitted $\mathfrak{H}_\varphi$.

b. The second equation has been derived from the r- and ϑ-components of (1). By means of a simple calculation it can be retransformed into the first Maxwellian equation, from which it was deduced originally. Let us introduce the substitution $\mathfrak{H} = \operatorname{rot}\mathfrak{A}$ into the first Maxwellian equation :

$$-\frac{c}{\mu}\mathfrak{E} - [\operatorname{rot}\mathfrak{B}, \operatorname{rot}\mathfrak{A}] = \varkappa\Delta\mathfrak{A}, \tag{3}$$

the φ-component of which gives our second equation

$$\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\,\mathfrak{B}_\varphi)\frac{1}{r}\frac{\partial}{\partial r}(r\mathfrak{A}_\varphi)-\frac{1}{r}\frac{\partial}{\partial r}(r\mathfrak{B}_\varphi)\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\,\mathfrak{A}_\varphi)=$$

$$=\varkappa\left(\Delta\mathfrak{A}_\varphi-\frac{\mathfrak{A}_\varphi}{r^2\sin^2\vartheta}\right), \tag{4}$$

supposing $\mathfrak{E}_\varphi = 0$, which is valid if the system be independent of the coordinate φ. From equation (4) $\mathfrak{A}_\varphi$ may be determined.

From the r-and ϑ-components of (3) we may express electric field strength and charge density. Let us form the divergence of the first Maxwellian equation:

$$\operatorname{div}\mathfrak{E} = -\frac{1}{c}\operatorname{div}[\mathfrak{v},\mathfrak{H}],$$

whence, by means of the substitution $\mathfrak{E} = \operatorname{grad}\Phi$, the following differential equation will be deduced for the potential Φ:

$$c\Delta\Phi = (\operatorname{rot}\mathfrak{H},\mathfrak{v}) - (\mathfrak{H},\operatorname{rot}\mathfrak{v}),$$

from which, if $\mathfrak{H}$ and $\mathfrak{v}$ are known, the electric potential may be calculated. In the present paper we do not occupy ourselves with the solution of the above equation, since observation of the local electric fields around the Sun as well as around the stars is very uncertain and offers no data by which the correctness of the theory could be controlled.

c. The third equation is nothing but the φ-component of the hydrodynamical equation of motion; from it $\mathfrak{v}_\varphi$ may be determined:

$$\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\,\mathfrak{B}_\varphi)\frac{1}{r}\frac{\partial}{\partial r}(r\mathfrak{v}_\varphi)-\frac{1}{r}\frac{\partial}{\partial r}(r\mathfrak{B}_\varphi)\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\,\mathfrak{v}_\varphi)-$$

$$-\frac{1}{\varrho}\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\,\mathfrak{A}_\varphi)\frac{1}{r}\frac{\partial}{\partial r}(r\mathfrak{H}_\varphi)+\frac{1}{\varrho}\frac{1}{r}\frac{\partial}{\partial r}(r\mathfrak{A}_\varphi)\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\,\mathfrak{H}_\varphi)=$$

$$=\nu\left(\Delta\mathfrak{v}_\varphi-\frac{\mathfrak{v}_\varphi}{r^2\sin^2\vartheta}\right). \tag{5}$$

d. We have to deduce another equation for $\mathfrak{B}_\varphi$, vector potential of the meridional currents. If ϱ = constant:

$$\mathfrak{v} = \operatorname{rot}\mathfrak{B};$$

this offers a relation among $\mathfrak{v}_r$, $\mathfrak{v}_\vartheta$ and $\mathfrak{B}_\varphi$, if the system is independent of the coordinate φ. Let us take its rotation:

$$\Delta\mathfrak{B} = -\operatorname{rot}\mathfrak{v} = -\mathfrak{c},$$

the φ-component of which is:

$$\Delta\mathfrak{B}_\varphi - \frac{\mathfrak{B}_\varphi}{r^2\sin^2\vartheta} = -\mathfrak{c}_\varphi. \tag{6a}$$

For $\mathfrak{c}_\varphi$ as a newly introduced unknown function we deduce a differential equation from the equation of motion. Let us take the rotation of the equation of motion :

$$\text{rot}\ [\text{rot}\ \mathfrak{B}, \mathfrak{c}] - \text{rot}\ [\text{rot}\ \mathfrak{A}, \text{rot}\ \mathfrak{H}] = \nu \Delta\, \mathfrak{c},$$

the φ-component of this is :

$$r \frac{\partial}{\partial r} \left(\frac{\mathfrak{v}_\varphi}{r \sin \vartheta} \right)^2 \sin \vartheta \cos \vartheta - \frac{\partial}{\partial \vartheta} \left(\frac{\mathfrak{v}_\varphi}{r \sin \vartheta} \right)^2 \sin^2 \vartheta -$$

$$- r \frac{\partial}{\partial r} \left(\frac{\mathfrak{H}_\varphi}{r \sin \vartheta} \right)^2 \sin \vartheta \cos \vartheta - \frac{\partial}{\partial \vartheta} \left(\frac{\mathfrak{H}_\varphi}{r \sin \vartheta} \right)^2 \sin^2 \vartheta =$$

$$= \nu \left(\Delta\, \mathfrak{c}_\varphi - \frac{\mathfrak{c}_\varphi}{r^2 \sin^2 \vartheta} \right). \tag{6b}$$

The equations (6*a*) and (6*b*) may be contracted into an only biharmonic equation, but as we shall see later, such transformations do not promote the solution of the equations.

For an entire discussion of the problem, besides the above equations, we need the equation of the potential of the gravitation, which we obtain by taking the divergence of the equation of motion :

$$\text{div}\,(\mathfrak{v}, \text{grad})\, \mathfrak{v} - \text{div}\,(\mathfrak{H}, \text{grad})\, \mathfrak{H} = - 4\pi G \varrho - \frac{1}{\varrho} \Delta P. \tag{7}$$

In what follows we assume that the star is spherical, the density constant ; therefore the solution of the above differential equation has no importance.

3. *Solution of the Basic Equations*

We solve the basic equations with the method of successive approximation. In assuming starting-values for $\mathfrak{B}_\varphi$ and $\mathfrak{H}_\varphi$ and in writing the same values into the differential equation which govern $\mathfrak{v}_\varphi$ and $\mathfrak{A}_\varphi$, we solve them and obtain an approximative expression for $\mathfrak{v}_\varphi$ and $\mathfrak{A}_\varphi$. In the second step we write these approximative expressions into the differential equations governing $\mathfrak{B}_\varphi$ and $\mathfrak{H}_\varphi$, obtaining approximative values for these functions. By means of conveniently chosen starting-values good approximation may be obtained even after an only step (if the starting-values for $\mathfrak{B}_\varphi$ and $\mathfrak{H}_\varphi$ have been so chosen that they may be also approximate values).

3.1. Development of $\mathfrak{A}_\varphi$.

Let us assume as starting-value for $\mathfrak{B}_\varphi$ that

$$\mathfrak{B}_\varphi = \beta \psi_2 P_2^{(1)}. \tag{8}$$

It is easily comprehended that this function may fulfil the boundary conditions. For ψ_2, or in general form ψ_l, the solution of the following differential equation is :

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi_l}{dr} \right) + \left(\lambda^2 - \frac{l(l+1)}{r^2} \right) \psi_l = 0. \tag{9}$$

If λ has been chosen so that the first zero of ψ_2 may be on the surface of the star, (8) will fulfil the boundary conditions of $\mathfrak{B}_\varphi$.

From the theory of the Bessel functions it is known that[1]

$$\psi_l = (\lambda r)^{-1/2} J_{l+1/2}(\lambda r),$$

where $J_{l+1/2}$ is a cylinder function of the order $l + 1/2$. Furthermore it is known that $J_{l+1/2}$, if l is an integer, may be expressed by a polynome, thus e. g. in the cases $l = 1$ and $l = 2$:

$$\psi_1 = \frac{\sin \lambda r}{\lambda^2 r^2} - \frac{\cos \lambda r}{\lambda r},$$

$$\psi_2 = \left(\frac{3}{\lambda^3 r^3} - \frac{1}{\lambda r}\right)\sin \lambda r - \frac{3}{\lambda^2 r^2} \cos \lambda r,$$

and the first zero of these functions are:

$$\lambda = \frac{4,4934}{r_1}, \quad \text{if} \quad l = 1,$$

$$\lambda = \frac{5,7608}{r_1}, \quad \text{if} \quad l = 2,$$

where r_1 is the stellar radius.

If we incidentally denote the left-hand side of (4) with $Q(\mathfrak{A}_\varphi)$, we may write:

$$\Delta \mathfrak{A}_\varphi - \frac{\mathfrak{A}_\varphi}{r^2 \sin^2 \vartheta} = \frac{1}{\varkappa} Q(\mathfrak{A}_\varphi). \tag{10}$$

According to our theory, published in a preceding paper[2], $\varkappa$ is, in order of magnitude, equal to ν, consequently it is a great number. Hence the convergence of the subsequent series is assured:

$$\mathfrak{A}_\varphi = \sum_{n=0}^{\infty} \left(\frac{1}{\varkappa}\right)^n A_n. \tag{11}$$

As Q, in its quality of operator, linearly contains $\mathfrak{A}_\varphi$, we may write:

$$Q(\mathfrak{A}_\varphi) = \sum_{n=0}^{\infty} \left(\frac{1}{\varkappa}\right)^n Q(A_n). \tag{12}$$

If we write (11) and (12) into (10), we shall obtain the following recurrence formula:

$$\Delta A_n - \frac{A_n}{r^2 \sin^2 \vartheta} = Q(A_{n-1}), \tag{13}$$

[1] Jahnke—Emde: Funktionentafeln, 1928, p. 91.

[2] Mitt. der Sternwarte Budapest, No. 26., p. 9.

where the right-hand side means the following expression:

$$Q(A_{n-1}) = \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} (\sin \vartheta \, \mathfrak{B}_\varphi) \frac{1}{r} \frac{\partial}{\partial r} (r A_{n-1}) - $$

$$- \frac{1}{r} \frac{\partial}{\partial r} (r \mathfrak{B}_\varphi) \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} (\sin \vartheta A_{n-1}).$$

A_0 results from the following differential equation:

$$\Delta A_0 - \frac{A_0}{r^2 \sin^2 \vartheta} = 0,$$

which in the interior of the star gives a value free of singularity only if

$$A_0 = H_0 r \sin \vartheta = H_0 r P_1^{(1)}, \tag{14}$$

more generally:

$$A_0 = \sum_{n=1}^{\infty} H_{n-1} r_n P_n^{(1)},$$

where $H_0, H_1, \ldots, H_n$ are arbitrary constants; in the centre of the star, however, the magnetic field must needs be homogeneous, therefore the condition $H_0 \neq 0$ has to exist.

In what follows the computations will be performed by means of the vector potential afforded by formula (14), for it gives sufficient precision, as it will be seen later. Therefore A_0 is nothing but the internal magnetic vector potential of a sphere of homogeneus magnetization. The lines of force run parallel to the axis of rotation. The value of field intensity equals H_0 in every point of the space.

By means of a simple computation we obtain that the differential equation for A_1, in virtue of (13), is:

$$\Delta A_1 - \frac{A_1}{r^2 \sin^2 \vartheta} = - \frac{6 \lambda \beta H_0}{5} \psi_1 P_1^{(1)} + \frac{4 \lambda \beta H_0}{5} \psi_3 P_3^{(1)}.$$

In computing the right-hand side we have availed ourselves of the recurrence formulae relative to the functions ψ_l.[1] We seek for a solution in the following form:

$$A_1 = a_1 P_1^{(1)} + a_3 P_3^{(1)},$$

for a_1 and a_3 these differential equations may be deduced:

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{da_1}{dr} \right) - \frac{2 a_1}{r^2} = - \frac{6 \lambda \beta H_0}{5} \psi_1,$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{da_3}{dr} \right) - \frac{12 a_3}{r^2} = \frac{4 \lambda \beta H_0}{5} \psi_3.$$

[1] Vid. Appendix 1.

The solution are obtained by the variations of the parameters :

$$a_1 = \frac{2\alpha H_0}{5} r(\psi_0 + \psi_2) = \frac{6\beta H_0}{5\lambda}\psi_1,$$

$$a_3 = \frac{4\alpha H_0}{35} r(\psi_2 + \psi_4) = \frac{4\beta H_0}{5\lambda}\psi_3,$$

where the recurrence formulae relative to ψ_l have been used repeatedly.

If we write the above expression and (14) into (11), we obtain $\mathfrak{A}_\varphi$ expanded in a series of the powers of $1/\varkappa$, stopping at the linear terms :

$$\mathfrak{A}_\varphi = H_0 r P_1^{(1)} + \frac{2H_0\beta}{5\varkappa\lambda}(3\psi_1 P_1^{(1)} - 2\psi_3 P_3^{(1)}). \tag{15}$$

At this passage we remark that this expression produces only the curls of the internal magnetic field, not fulfilling other conditions of the field ; it is later that we shall be engaged in evaluating these.

3.2. Development of $\mathfrak{v}_\varphi$.

The differential equation affording $\mathfrak{v}_\varphi$ is analogous to that of $\mathfrak{A}_\varphi$. In the case $\mathfrak{H}_\varphi = 0$ these two equations will have entirely identical forms. But as we saw in our previous paper,[1] such a simplification of the problem does not lead to any physically interpretable result.

$\mathfrak{H}_\varphi$ cannot be identically zero in the interior of the star. For this reason $\mathfrak{v}_\varphi$ and $\mathfrak{A}_\varphi$ can never be of identical distribution. Let us assume for $\mathfrak{H}_\varphi$, similarly to what was done for $\mathfrak{B}_\varphi$, the following starting-value :

$$\mathfrak{H}_\varphi = a\psi_2 P_2^{(1)}, \tag{16}$$

which, apart from the coefficient, is identical with the initial value of $\mathfrak{B}_\varphi$. This circumstance is motivated by the fact that the boundary conditions in relation to both functions are identical.

Let as write (16) into the left-hand side of (5) and substitute for $\mathfrak{v}_\varphi$ the following series :

$$\mathfrak{v}_\varphi = \sum_{n=0}^{\infty}\left(\frac{1}{\nu}\right)^n V_n, \tag{17}$$

which is convergent, since ν, coefficient of the turbulent viscosity, is a very large number.

For V_0 and V_1 we obtain this differential equation :

$$\Delta V_0 - \frac{V_0}{r^2\sin^2\vartheta} = 0,$$

whence

$$V_0 = \omega_0 r \sin\vartheta$$

and

$$\Delta V_1 - \frac{V_1}{r^2\sin^2\vartheta} = \frac{6}{5}\left(\omega_0\beta + \frac{\alpha H_0}{\varrho}\right)\lambda\psi_1 P_1^{(1)} - \frac{4}{5}\left(\omega_0\beta + \frac{\alpha H_0}{\varrho}\right)\lambda\psi_3 P_3^{(1)},$$

[1] Mitt. der Sternwarte Budapest, No. 32.

the solution of this is entirely similar to that of A_1:

$$V_1 = \frac{2}{5\lambda}\left(\omega_0\beta + \frac{\alpha H_0}{\varrho}\right)(3\psi_1 P_1^{(1)} - 2\psi_3 P_3^{(1)});$$

hence

$$\mathfrak{v}_\varphi = \omega_0 r P_1^{(1)} + \frac{2}{5\lambda}\frac{1}{r}\left(\omega_0\beta + \frac{\alpha H_0}{\varrho}\right)(3\psi_1 P_1^{(1)} - 2\psi_3 P_3^{(1)}). \tag{18}$$

In the case $\mathfrak{H}_\varphi = 0$, i. e. $\alpha = 0$, this value will be completely analogous to that of $\mathfrak{A}_\varphi$. Consequently our result is in correspondence with the case $\mathfrak{A}_\varphi = \mathfrak{v}_\varphi$, obtained in the previous paper.

3.3. Development of $\mathfrak{H}_\varphi$

In the above remarks we have determined the functions $\mathfrak{A}_\varphi$ and $\mathfrak{v}_\varphi$, developing them into the series of the powers of $1/\varkappa$ and $1/\nu$. Equations (15) and (18) give the expression of them up to linear terms. For the calculations we have assumed the subsequent starting-values in reference to $\mathfrak{B}_\varphi$ and $\mathfrak{H}_\varphi$:

$$\mathfrak{B}_\varphi = \beta\psi_2 P_2^{(1)} \quad \text{and} \quad \mathfrak{H}_\varphi = \alpha\psi_2 P_2^{(1)}.$$

In the present chapter we are seeking the solution for $\mathfrak{H}_\varphi$, in the subsequent one for $\mathfrak{B}_\varphi$, by availing ourselves of (15) and (18). In this way we obtain $\mathfrak{H}_\varphi$ in first approximation.

The two first members on the left-hand side of (2) contain the second order products of $\mathfrak{B}_\varphi$ and $\mathfrak{H}_\varphi$; consequently, being quantities of second order, they may be neglected in first approximation. Now let us substitute into the third and fourth members the functions $\mathfrak{A}_\varphi$ and $\mathfrak{v}_\varphi$, obtained just now by means of a development in the terms of the powers of $1/\varkappa$ and $1/\nu$ up to the linear terms:

$$\Delta\mathfrak{H}_\varphi - \frac{\mathfrak{H}_\varphi}{r^2\sin^2\vartheta} =$$

$$= \frac{H_0\omega_1\lambda}{\varkappa}\left[6r\frac{d}{dr}\left(\frac{\psi_1}{r}\right)P_1 P_1^{(1)} - 4r\frac{d}{dr}\left(\frac{\psi_3}{r}\right)P_1 P_3^{(1)} + 4\frac{\psi_3}{r}\sin\vartheta\frac{d}{d\vartheta}\left(\frac{P_3^{(1)}}{\sin\vartheta}\right)P_1^{(1)}\right],$$

where

$$\omega_1 = \frac{2}{5}\,\frac{\omega_0\beta\varrho + H_0\alpha}{\nu\varrho}.$$

Furthermore the following relations exist[1]:

$$r\frac{d}{dr}\left(\frac{\psi_1}{r}\right) = -\lambda\psi_2,$$

$$r\frac{d}{dr}\left(\frac{\psi_3}{r}\right) = \frac{2\lambda}{7}\psi_2 - \frac{5\lambda}{7}\psi_4,$$

[1] Vid. Appendix 1.

$$P_1 P_1^{(1)} = \sin\vartheta \cos\vartheta = \frac{1}{3} P_2^{(1)},$$

$$P_1 P_3^{(1)} = \frac{4}{7} P_2^{(1)} + \frac{3}{7} P_4^{(1)},$$

$$\sin\vartheta \frac{d}{d\vartheta}\left(\frac{P_3^{(1)}}{\sin\vartheta}\right) = -\sin^2\vartheta \frac{d^2 P_3}{d(\cos\vartheta)^2} = -P_3^{(2)}$$

and

$$P_1^{(1)} P_3^{(2)} = \frac{20}{7} P_2^{(1)} - \frac{6}{7} P_4^{(1)};$$

by way of substitutions executed with these values we obtain :

$$\Delta \mathfrak{H}_\varphi - \frac{\mathfrak{H}_\varphi}{r^2 \sin^2\vartheta} = -\frac{12 H_0 \omega_1}{35 \varkappa}(5\psi_2 P_2^{(1)} - 2\psi_4 P_4^{(1)}), \tag{19}$$

hence

$$\mathfrak{H}_\varphi = \frac{12 H_0}{35} \cdot \frac{\omega_0 \beta \varrho + H_0 a}{\varkappa \nu \varrho}(5\psi_2 P_2^{(1)} - 2\psi_4 P_4^{(1)}. \tag{20}$$

3.4. *Development of* $\mathfrak{B}_\varphi$

In order to compute the vector potential of the meridional current we have, first of all, to determine the vector $\mathfrak{c}_\varphi$ by means of the solution of equation (6*b*). If we write equations (18) and (16) into (6*b*), we obtain by neglecting the second order qualities :

$$\Delta \mathfrak{c}_\varphi - \frac{\mathfrak{c}_\varphi}{r^2 \sin^2\vartheta} = \frac{1}{\nu}\left[r \frac{\partial}{\partial r}\left(\frac{\mathfrak{v}_\varphi}{r\sin\vartheta}\right)^2 \sin\vartheta\cos\vartheta - \frac{\partial}{\partial\vartheta}\left(\frac{\mathfrak{v}}{r\sin\vartheta}\right)^2 \sin^2\vartheta\right.$$

and, considering only the linear terms, $(\mathfrak{v}_\varphi / r \sin\vartheta)^2$ will be :

$$\left(\frac{\mathfrak{v}_\varphi}{r\sin\vartheta}\right)^2 = \omega_0 + \frac{2\omega_0\omega_1}{\lambda}\left[\frac{3\psi_1 - 2\psi_3}{r} P_0 - 10 \frac{\psi_3}{r} P_3\right],$$

by way of substitutions performed with these values, after a certain modification we obtain :

$$\Delta \mathfrak{c}_\varphi - \frac{\mathfrak{c}_\varphi}{r^2 \sin^2\vartheta} = \frac{6\omega_0\omega_1}{7\nu}(5\psi_2 P_2^{(1)} - 2\psi_4 P_4^{(1)}).$$

The solution of this equation is :

$$\mathfrak{c}_\varphi = \frac{6\omega_0}{7} \frac{\omega_0 \beta \varrho + H_0 a}{\nu^2 \varrho}(5\psi_2 P_2^{(1)} - 2\psi_4 P_4^{(1)}). \tag{21}$$

Finally, by help of $\mathfrak{c}_\varphi$ just determined, let us express $\mathfrak{B}_\varphi$ from differential equation (6a). Writing (21) into (6a) we obtain

$$\Delta\mathfrak{B}_\varphi - \frac{\mathfrak{B}_\varphi}{r^2 \sin^2 \vartheta} = -\frac{6\omega_0}{7} \frac{\omega_0 \beta \varrho + H_0 \alpha}{\nu^2 \varrho} (5\psi_2 P_2^{(1)} - 2\psi_4 P_4^{(1)}),$$

the solution of which is

$$\mathfrak{B}_\varphi = \frac{6\omega_0}{7} \frac{\omega_0 \beta \varrho + H_0 \alpha}{\nu^2 \lambda^2 \varrho} (5\psi_2 P_2^{(1)} - 2\psi_4 P_4^{(1)}). \tag{22}$$

4. Discussion of the Basic Equations

In the preceding chapter we have determined the four wanted functions in first approximation. The correctness of solutions is shown by the fact that, after the integration, we have reobtained the starting-values for the functions $\mathfrak{H}_\varphi$ and $\mathfrak{B}_\varphi$. The functions obtained by means of integration, however, contain also terms of higher order, which may be interpreted in such a way that the functions are expressed in better approximations than the starting-values. The further proceeding would be an iterated integration of the differential equations with these new functional values. In the present paper, however, we do not perform this »second approximation«. For in the further approximations neither the uniform density nor the spherical shape of the star can be applied.

In the present chapter we are going to indicate some important properties of $\mathfrak{A}_\varphi$, $\mathfrak{v}_\varphi$, $\mathfrak{H}_\varphi$ and $\mathfrak{B}_\varphi$.

4.1. Relations between the parameters

The condition that the first order terms of the functions $\mathfrak{H}_\varphi$ and $\mathfrak{B}_\varphi$ should correspond to their starting-values, is the necessity of the agreement of the coefficients. The starting-values of the functions $\mathfrak{H}_\varphi$ and $\mathfrak{B}_\varphi$ were:

$$\mathfrak{B}_\varphi = \beta \psi_2 P_2^{(1)} \quad \text{and} \quad \mathfrak{H}_\varphi = \alpha \psi_2 P_2^{(1)},$$

whereas the terms corresponding to their integrated values are:

$$\mathfrak{B}_\varphi = \frac{30\omega_0}{7} \frac{\omega_0 \beta \varrho + H_0 \alpha}{\nu^2 \lambda^4 \varrho} \psi_2 P_2^{(1)},$$

$$\mathfrak{H}_\varphi = \frac{30 H_0}{7} \frac{\omega_0 \beta \varrho + H_0 \alpha}{\varkappa \nu \lambda^2 \varrho} \psi_2 P_2^{(1)},$$

i. e.

$$\beta = \frac{30\omega_0}{7} \frac{\omega_0 \beta \varrho + H_0 \alpha}{\nu^2 \lambda^4 \varrho} \tag{23}$$

and

$$\alpha = \frac{30 H_0}{7} \frac{\omega_0 \beta \varrho + H_0 \alpha}{\varkappa \lambda^2 \varrho} \tag{24}$$

must be doubtless.

As these two equations are linear and homogeneous in α and β, only the ratio α/β may be determined from them, if the determinant of the system is equal to zero. The ratio α/β is obtained if (24) is divided by (23):

$$\frac{\alpha}{\beta} = \frac{H_0 \nu \lambda^2}{\omega_0 \varkappa} \tag{25}$$

and the vanishing of the determinant is expressed by the following equation:

$$\frac{\omega_0^2}{\nu} + \frac{H_0^2 \lambda^2}{\varkappa \varrho} = \frac{7 \nu \lambda^4}{30}. \tag{26}$$

According to (25) the independent parameters are α, $\varkappa$, ν, ω_0 and H_0; according to (26) there is a relation between these, i. e. the number of the independent parameters diminishes by one: they are either α, ν, ω_0 and H_0 or $\varkappa$, ν, ω_0 and H_0. But in the course of the further discussion it results that the number of the parameters can be still reduced. We remark, however, even at this passage that within the limits of the present theory we can expect no connection between ω_0 and H_0. These have to be considered as independent parameters.

4.2. Distribution of the magnetic field

The solution of (2) was:

$$\mathfrak{A}_\varphi = H_0 r P_1^{(1)} + \frac{2\beta H_0}{5\varkappa\lambda}(3\psi_1 P_1^{(1)} - 2\psi_3 P_3^{(1)}).$$

This expression produces only the curls of the magnetic field, its part free of curls is obtained by means of solving the equation

$$\Delta \mathfrak{A}_\varphi - \frac{\mathfrak{A}_\varphi}{r^2 \sin^2 \vartheta} = 0,$$

whence $\mathfrak{A}_\varphi = r^n P_n^{(1)}$ and $\mathfrak{A}_\varphi = P_n^{(1)}/r^{n+1}$. So let us complete $\mathfrak{A}_\varphi$ with terms like these, and we shall obtain:

$$\mathfrak{A}_\varphi = H_0 \left\{ \left(r + \frac{6\beta}{5\lambda\varkappa} \psi_1 \right) P_1^{(1)} + \left(h r^3 - \frac{4\beta}{5\lambda\varkappa} \psi^3 \right) P_3^{(1)} \right\}.$$

The external field consists only of terms free of rotation:

$$\mathfrak{A}_\varphi = H_0 \left\{ \frac{m_1}{r^2} P_1^{(1)} + \frac{m_3}{r^4} P_3^{(1)} \right\},$$

where m_1 and m_3 are constant. $\mathfrak{A}_\varphi$ and rot $\mathfrak{A}_\varphi$ have to pass continuously the stellar surface taken by us for approximately spherical. Consequently the following expressions must hold on the one hand in the internal field:

$$\mathfrak{H}_r = H_0 \left\{ \left(1 + \frac{6\beta}{5\lambda\varkappa} \frac{\psi_1}{r} \right) 2P_1 + \left(h r^2 - \frac{4\beta}{5\lambda\varkappa} \frac{\psi_3}{r} \right) 12 P_3 \right\}$$

$$\mathfrak{H}_\vartheta = -H_0 \left\{ \left[2 + \frac{6\beta}{5\varkappa} \left(\frac{2}{3}\psi_0 - \frac{1}{3}\psi_2 \right) \right] P_1^{(1)} + \left[4 h r^2 - \frac{4\beta}{5\varkappa} \left(\frac{4}{7}\psi_2 - \frac{3}{7}\psi_4 \right) \right] P_3^{(1)} \right\},$$

on the other hand in the external field :

$$\mathfrak{H}_r = H_0 \left\{ \frac{2m_1}{r} P_1 + \frac{12m_3}{r^5} P_3 \right\},$$

$$\mathfrak{H}_\vartheta = H_0 \left\{ \frac{m_1}{r} P_1^{(1)} + \frac{3m_3}{r^5} P_3^{(1)} \right\}.$$

The conditions of mathematical continuity are :

$$1 + \frac{2}{5} \frac{\beta}{\varkappa} \psi_0(\lambda r_*) = \frac{m_1}{r_*^3},$$

$$hr_*^2 - \frac{4}{35} \frac{\beta}{\varkappa} \psi_4(\lambda r_*) = \frac{m_3}{r_*^5},$$

$$1 + \frac{2}{5} \frac{\beta}{\varkappa} \psi_0(\lambda r_*) = -\frac{1}{2} \frac{m_1}{r_*^3},$$

$$4hr_*^2 + \frac{12}{35} \frac{\beta}{\varkappa} \psi_4(\lambda r_*) = -\frac{3m_3}{r_*^5}$$

from which it follows that m_1 is equal to zero, i. e. the external dipole field is identically zero, furthermore :

$$\frac{\beta}{\varkappa} = 28{,}865 ; \quad \frac{m_3}{r_*^5} = -0{,}6631 ; \quad hr_*^2 = 0.$$

Consequently the external field is only a pole of third order i. e. an *octopole-type field*. But this cannot be looked upon by us as definitive result of the theory ; from it may be solely concluded that induced electrical currents taken into consideration are not sufficient for the interpretation of the permanent stellar field. Such a field can be deduced only by the aid of a particular theory, for instance if it is supposed that also a charge-transport arises along with a mass-transport. Such a phenomenon may be theoretically derived from the electron diffusion in the non-conservative force-field (whirl-field).

By making use of the above relations we obtain as internal potential :

$$\mathfrak{A}_\varphi = H_0 r_* \left\{ \left[\frac{r}{r_*} + 133{,}03 \psi_1(\lambda r) \right] P_1^{(1)} - 199{,}54 \psi_3(\lambda r) P_3^{(1)} \right\}; \qquad (25a)$$

and as external potential :

$$\mathfrak{A}_\varphi = -33{,}12 H_0 \frac{r_*^5}{r^4} P_3^{(1)}. \qquad (25b)$$

On principle the discussion affords some significant meanings.

1. Within its present extent the theory is not suitable to the interpretation of the permanent stellar field. For this purpose further theories are required.

2. $\beta/\varkappa$ is constant, whence it follows that in the case of infinite conductivity ($\varkappa \to 0$) the meridional current will vanish ($\beta \to 0$).

4.3. Problem of the permanent magnetic field

From what has been stated it follows that the present theory is not suitable for the deduction of the permanent magnetic fields of the stars and those of the Sun. Instead of the homogeneous field H_0, introduced into (14), — a field free of divergence and curls —, we should have to assume a vortical field. Nevertheless to introduce a vortical field is impossible within the limits of the present theory. By the aid of a new hypothesis we have to deduce it from the electric currents running about the axis. According to our present conception non-conservative electron diffusion arising in a field of vorticity may originate such currents. The phenomenon resembles the change of energy, arising in consequence of internal friction. The charge of energy of the ions and that of the electrons are different from each other, therefore also their distribution of velocity will be different; consequently in the separate parts of the space the velocity of the electrons in their rotation about the stellar axis is greater than that of the ions. This phenomenon will be equivalent to an electrical current, which gives rise to a magnetic field whithout the formation of simultaneous charge of space.

Let us suppose that electron diffusion is not influenced by any other electromagnetic phenonena (charge of space, induction) and that charge diffusion is proportional to $(\mathfrak{v}, \text{grad})\,\mathfrak{v}$. In this case the differential equation for the vector potential of the magnetic field is:

$$\Delta \mathfrak{A}_\varphi - \frac{\mathfrak{A}_\varphi}{r^2 \sin^2 \vartheta} = C\,\omega\,\psi_1 P_1^{(1)},$$

which leads to the same result as (13). Accordingly the permanent dipole field cannot be expressed even by such a modification of the theory. Therefore we have to conclude that dipole field cannot be deduced from the first approximation elaborated at present, but, in order to obtain it, we must introduce also the distribution of density and temperature.

4.4 Toroidal magnetic field

The component of the magnetic field, perpendicular to the meridional plane, is given by (20). It produces a toroidal field. The lines of force of this field are circles concentric with the axis of rotation and parallel to the equator.

In what follows we examine the formula of $\mathfrak{H}_\varphi$ given by (16), in accordance with this:

$$\mathfrak{H}_\varphi = \frac{H_0 \lambda^2 \beta \nu}{\omega_0 \varkappa} \psi_2 P_2^{(1)},$$

or after making use of the relations

$$\frac{\beta}{\varkappa} = 28{,}865 \quad \text{and} \quad \lambda r_* = 5{,}7608$$

we obtain:

$$\mathfrak{H}_\varphi = 957{,}9 \frac{H_0 \nu}{\omega_0 r_*^2} \psi_2 P_2^{(1)}. \tag{29}$$

By virtue of (2), the subsequent differential equation refers to the external field :

$$\Delta \mathfrak{H}_\varphi - \frac{\mathfrak{H}_\varphi}{r^2 \sin^2 \vartheta} = 0.$$

We have to seek those solutions of the above equation which are zero on the stellar surface. It is easily intelligible that this condition can be satisfied in the whole external field only by the trivial solution $\mathfrak{H}_\varphi = 0$, i. e. the external field of $\mathfrak{H}_\varphi$ is identically zero, hence the internal field must form an utterly closed system in the interior of the star. And this is also clear, for the lines of force, being concentric circles, are closed in the stellar interior. The toroidal field intensity $\mathfrak{H}_\varphi$ cannot be observed ; its existence, however, as shown in the preceding paper, is necessary, if the star rotates and has a magnetic field.

A toroidal magnetic field takes rise from meridional electric currents, for according to the Maxwellian equations :

$$\frac{i}{c} = \operatorname{rot} \mathfrak{H}_\varphi.$$

From the fact that the expression of $\mathfrak{H}_\varphi$ is analogous to that of $\mathfrak{B}_\varphi$, it follows that the family of trajectories of the electric streamlines is identical to that of meridional mass-currents.

4.5. *Distribution of the velocity field*

Introducing relations (25), (26) and (28) into (18), we obtain

$$\mathfrak{v}_\varphi = \omega_0 r P_1^{(1)} + \frac{7\lambda^2 \beta \nu}{75\omega_0} (3\psi_1 P_1^{(1)} - 2\psi_3 P_3^{(1)}).$$

If we introduce trigonometric functions instead of spherical ones, we get by a simple computation the expression of the distribution of angular velocity :

$$\omega = \omega_0 + \frac{7\lambda^2 \beta \nu}{5\omega_0} \frac{\psi_2}{r^2} - \frac{7\lambda^3 \beta \nu}{5\omega_0} \frac{\psi_3}{r} \cos^2 \vartheta. \tag{30a}$$

On the stellar surface, approximately taken for spherical, $\psi_2 = 0$ and $\psi_3 = 0{,}166$; which values, if written in, give the surface angular velocity distribution :

$$\omega = \omega_0 - \frac{1282{,}5 \beta \nu}{\omega_0 r_*^4} \cos^2 \vartheta. \tag{30b}$$

Substituting $\vartheta = 90°$ we get : ω_0 means the angular velocity along the equator (on the surface), whereas the angular velocity at the poles is :

$$\omega_p = \omega_0 - \frac{1282{,}5 \varkappa \nu}{\omega_0 r_*^4},$$

i. e. the polar angular velocity is less than the equatorial one. As the polar angular velocity cannot be zero or a negative value, the subsequent inequality must hold:

$$\omega_0^2 > 1282,5\,\frac{\varkappa\nu}{r_*^4},$$

which, in the case of given parameters $\varkappa$ and ν, affords the inferior limit or ω_0. The value of the angular velocity is highest in the centre of the star:

$$\omega_c = \omega_0 + \frac{89,41\,\varkappa\nu}{\omega_0 r_*^4}.$$

Expressions of velocity distribution may be essentially simplified, if the magnetic field strength is not excessively great. For in accordance with (26)

$$\nu = \frac{15}{7\cdot(5,7608)^2}\left(\frac{H_0^2}{\varkappa\varrho} + \sqrt{\frac{H_0^4}{\varkappa^2\varrho^2} + \frac{14}{15}\omega_0^2}\right). \tag{31}$$

If the magnetic field is very weak, $H_0^2/\varkappa\varrho$ may be negligible beside ω_0, and then

$$\nu = \sqrt{\frac{30}{7}\,\frac{\omega_0 r_*^2}{(5,7608)^2}} = 0,06238\,\omega_0 r_*^2, \tag{31a}$$

or let us indicate the equatorial angular velocity by means of $v_0 = \omega_0 r_*$, and then we may write

$$\nu = 0,06238\,v_0 r_*. \tag{31b}$$

This formula affords us a further relation between the parameters. It renders possible that the order of magnitude of ν may be determined by the aid of v_0 and r_*; by dint of this order determined, the question whether the internal friction has molecular or turbulent origin, will be settled. We are able to determine the coefficient of molecular viscosity from the state of the gas by the aid of molecular theory. According to this theory the kinematical viscosity is about 10^7, for instance on the surface of the Sun. As to the coefficient of turbulent viscosity, there is no acceptable theory at present. If we are capable of estimating the average size of eddies and the mean value of the velocity fluctuation, we obtain, by virtue of the formula $\nu \sim \overline{|\mathfrak{v}'|\,.\,|\mathfrak{l}'|}$ an approximative value for the coefficient of viscosity. According to computations of the kind the coefficient of viscosity on the surface of the Sun is of the order of magnitude of 10^{13}. Equation (31b) expresses an important connexion between turbulent viscosity and impulse momentum. In order to verify this significant result of the present theory, we shall still return to this question in what follows.

5. *Comparison with Solar Observations*

5.1. *Distribution of the angular velocity of the Sun*

Having computed the distribution of angular velocity on the surface, expressed by (30b) in relation to some star, let us substitute the solar radius ($r_\odot$) for r_*; thus

$$\omega = \omega_0 - \frac{1282,5}{\omega_0 r_\odot^4}\,\varkappa\nu\cos^2\vartheta,$$

which has identical form with that of the well-known empirical formula of interpolation :

$$\omega = \frac{2 \cdot 10^5}{r_\odot} - \frac{0,4 \cdot 10^5}{r_\odot} \cos^2 \vartheta .$$

By the comparison of the two formulae we obtain

$$\omega_0 = \frac{2,0 \cdot 10^5}{r_\odot}$$

and

$$\varkappa \nu \frac{1282,5}{\omega_0 r_\odot^3} = 0,4 \cdot 10^5 ;$$

whence

$$\frac{\varkappa \nu}{r_\odot} = \frac{0,8}{1282,5} 10^{10} r_\odot = 4,366 \cdot 10^{17} . \tag{32}$$

$\varkappa$ and ν may be determined from the connexions between the parameters as well as from the streamings on the solar surface. Such computations can be made by the aid of an adequate discussion of the meridional motions.

5.2. Meridional motions

The velocity of the meridional currents on the solar surface can be determined from the vector potential $\mathfrak{B}_\varphi$. According to (8) :

$$\mathfrak{B}_\varphi = \beta \psi_2 P_2^{(1)}$$

or if (28) is improved, then :

$$\mathfrak{B}_\varphi = 28,865 \varkappa \psi_2 P_2^{(1)} ; \tag{33}$$

from these the meridional components of the velocity are :

$$\mathfrak{v}_r = \frac{57,73}{r} \psi_2 P_2 ,$$

$$\mathfrak{v}_\vartheta = \frac{28,87 \lambda \varkappa}{5} (2 \psi_3 - 3 \psi_1) P_2^{(1)} .$$

If we suppose that the meridional drift of the sunspots gives the meridional motion of the photosphere, $\mathfrak{v}_\vartheta$ may be directly calculated from the meridional drift of the sunspots, viz.

$$\mathfrak{v}_\vartheta = r_\odot \dot{\vartheta} ,$$

where $\dot{\vartheta}$ is the angular velocity of the meridional drift :

$$\dot{\vartheta} = 2.10^{-9} \text{ radian per second,}$$

whence

$$\mathfrak{v}_\vartheta = 1,4 \cdot 10^2 \text{ cm/sec.}$$

The distribution of the velocity on the surface in accordance with the theory is:

$$\mathfrak{v}_\vartheta = 27{,}60 \frac{\varkappa}{r_\odot} P_2^{(1)},$$

from this, if $P_2^{(1)} = 1$ ($\vartheta = 45°$),

$$\mathfrak{v}_\vartheta = 27{,}60 \frac{\varkappa}{r_\odot} = 3{,}94 \cdot 10^{-10}\varkappa,$$

after comparing this value with the empirical one, gained before, we obtain :

$$\varkappa = 3{,}75 \cdot 10^{13}, \tag{34}$$

the conductivity will be $\sigma \sim 10^7$. The molecular conductivity may be determined from the degree of ionization, according to which $\sigma = 10^{12}$. Consequently the coefficient of conductivity, deduced theoretically, cannot be interpreted by the aid of the molecular theory, because the coefficient in question is five orders of magnitude larger than that obtainable from the molecular theory. A medium of such en exiguous conductivity has to be considered as an isolator. Certain suppositions of the theory of hydromagnetic turbulence indicate that in turbulence the mean charge transport of very large space elements is very small ; therefore turbulent electric conductivity may be intrôduced, which is several orders of magnitude smaller than molecular conductivity. From our discussion we may conclude that there must be hydromagnetic turbulence in the Sun and that the coefficient, obtained by means of (34), is one of the characteristics for such a turbulent state, viz. it is electric resistance in E. S. U.

From (32) and (34) also ν may be determined :

$$\nu = \frac{4{,}366}{3{,}57} r_\odot 10^4 = 1{,}556 \cdot 10^{15}. \tag{35}$$

This coefficient is two orders of magnitude larger than that deduced in another way from the turbulence of the solar surface. For the determination of ν we have taken into consideration also the turbulence in the interior of the Sun. Consequently we have to think this deviation arises from the circumstance that turbulence in the Sun's interior is more vigorous than that observable on this surface.

The coefficient of viscosity, however, may be determined, on the ground of (31*b*), also from the equatorial velocity and the radius :

$$\nu = 0{,}06238 \cdot 2 \cdot 7 \cdot 10^{10} \cdot 10^5 \sim 10^{15}. \tag{36}$$

This value shows a sufficiently good conformity with the above coefficient, determining empirically, from which circumstance we are permitted to conclude the correctness of the theory.

5.3. *Distribution of the magnetic field*

From our theory it has followed that the dipole momentum of the external magnetic field is zero in first approximation. According to the observations the Sun has a dipole field, which is, however, so weak that the determination of its structure is very uncertain. Exact measurements cannot be made but almost exclusively at polar areas, because under lower heliographic latitudes the determination of the mean magnetic field is practically impossible on account of the magnetic fluctuations of the sunspot zone. The theoretically obtained external magnetic field shows a certain deceptive similarity to the dipole field. For the magnetic polarity at the North pole is contrary to that at the South pole in the case of an octopole field as well as in that of a dipole one. The lines of force emerging from the polar areas are considerably curvated, and if $\vartheta \sim 60°$, they return into the Sun's interior. Therefore the polarity will be contrary here. From the environment of this place also another family of lines of force arises with identical polarity and having passed the plane of the equator, enters the surface in the area of $\vartheta = 120°$.

According to the recent magnetographic measurements[1] the Sun's polar field can be observed only from the pole to $\vartheta = 35°$ and from $\vartheta = 145°$ to $\vartheta = 180°$. In accordance with our theory the polarity of the field has to change both at about $\vartheta = 40°$ and $\vartheta = 140°$.

In what follows we may suppose that the magnetic field, observed in the polar area, derives from the theoretically deduced octopole field. If we assume that the field intensity at the poles be 1 gauss, we obtain:

$$\mathfrak{H}_r = -33{,}12 \cdot 12 \cdot H_0 = 1,$$

whence

$$H_0 = -0{,}002516. \tag{37}$$

According to (29) the horizontal component of the magnetic field is:

$$\mathfrak{H}_\varphi = 957{,}9 \frac{H_0 \nu}{\omega_0 r^2} \psi_2 P_2^{(1)},$$

which we may simplify making use of (31*a*):

$$\mathfrak{H}_\varphi = 59{,}74 H_0 \psi_2 P_2^{(1)},$$

or by the aid of the above value of H_0:

$$\mathfrak{H}_\varphi = 1{,}50\, \psi_2 P_2^{(1)}. \tag{38}$$

The maximum of the magnetic field in the Sun's interior, at the point given by the coordinates $r = 3{,}8$ and $\vartheta = 45°$, is:

$$H_{\varphi\,\max} = 0{,}4$$

consequently it is in order of magnitude equal to the polar field strength. So from the point of view of hydromagnetic turbulence coming to pass in the solar interior, it must have nearly as great an importance as that of the meridional field.

6. *Comparison with Stellar Observations*

By means of spectroscopic observations of the stars only ω_0 and H_0 can be determined. Therefore, making use of the observations, we may calculate only those parameters of our theory which depend on these two quantities.

[1] Ap. J. **121**, 348 (1955).

If we assume, however, that $\varkappa$ be a very great number also in the stars, in like manner as in the Sun, i. e. hydromagnetic turbulences exist, then (31b) will hold here too :

$$\nu = 0{,}06258\ v_{equ} r_{*}, \tag{39}$$

of the correctness of which we have convinced ourselves.

The correctness of (39), however, may be concluded from the comparison of the turbulence and rotational velocity of some stars type B, A, and F, on the ground of observations, or more properly, discussions of O. Struve, C. T. Elvey, K. O. Wright, Su Shu Huang and others. For provided that turbulent velocity is proportional to ν (or is a function of it), furthermore the observed velocity is equal to the equatorial one, and finally the radii of the stars in question may be considered as equal (in reality they are between 1—5☉), we are permitted to compare relation (39) with the observational results.

In the table below we have collected, using the literature at our disposal, the turbulent velocity and the rotational one of the stars type B, A, and F, and also those of the Sun :

Star	Type	v_{turb}	Litt.	v_{rot}	Litt.
ϱ Leo	BO	18	St, H	61,8	H
α CMa	AO	2,0	St	15,6	H
α Lyr!	AO	2,0	St	15,6	H
17 Lep	AO	67,0	St		
α Cyg	A2p	2,0	St	25,0	H
α Car	FO	3,5	H	15,6	H
ε Aur	F2	24,0	H	30,6	H
α Per	F5	7,5	H	15,6	H
α CMi	F5	4,0	H	15,6	H
α Cyg	F8	7,8	H		
δ CMa	F8	9,0	H		
Sun	GO	1,9		2,0	

(St = Struve, Ap. J. **79**,409 (1939) ; H = Su Shu Huang Ap. J. **118**,285 (1953).

By the aid this table fig. 1 has been drawn. As we may see, in it a certain relation between the two quantities appears decidedly, but because of the exiguous material of observation we get no satisfactory answer to its reality. It would be very necessary to complete the material with further data of those stars of fast rotation, the turbulent velocity of which is also known to us.

7. *Trajectories of the Velocity and Magnetic Field*

It is not unknown that the trajectories of a vector field, $\mathfrak{A} = \text{rot}\ \mathfrak{a}$, are given by the following differential equation :

$$\left[\text{rot}\ \mathfrak{a},\ \frac{d\mathfrak{r}}{ds}\right] = 0.$$

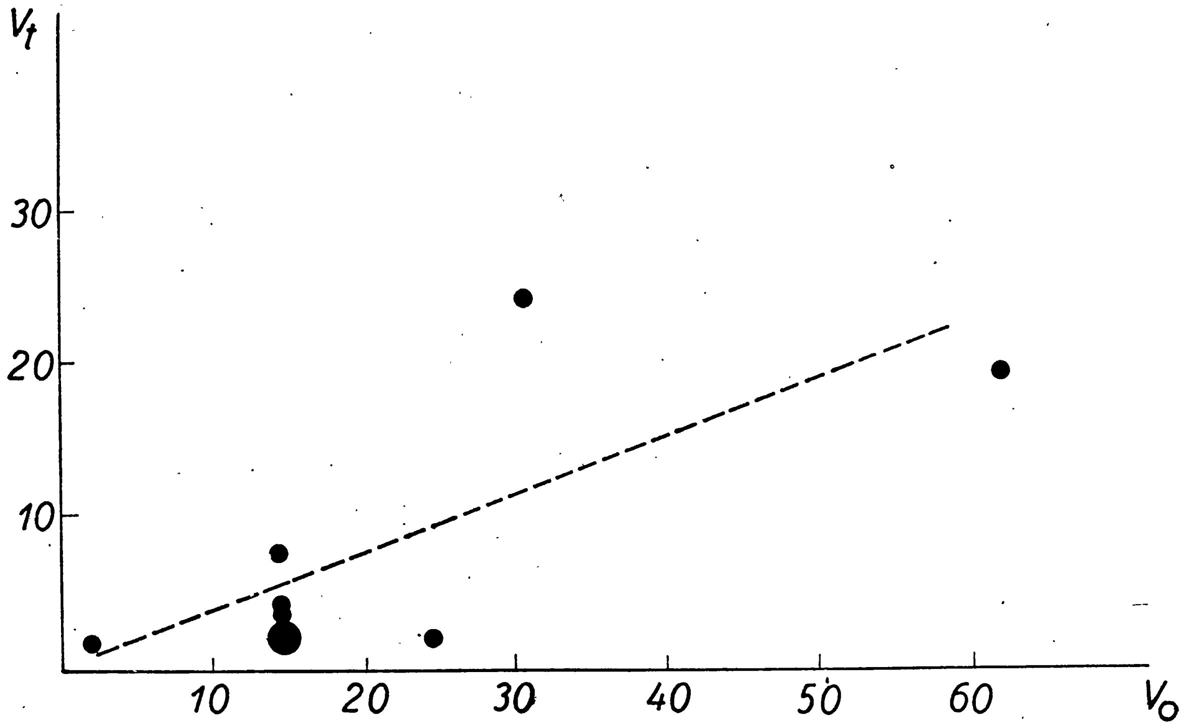

Fig. 1.

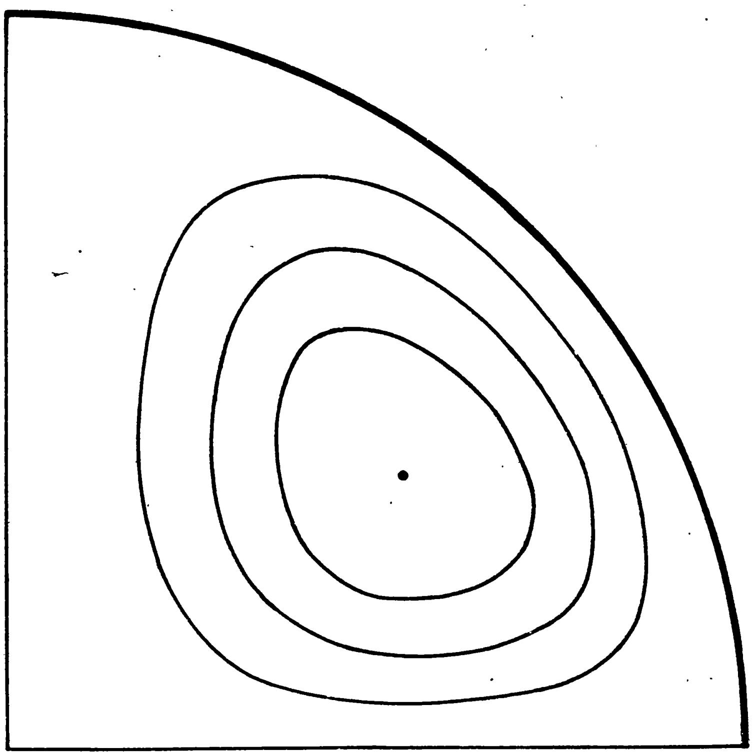

Fig. 2

If only the φ-component of the vector potential is different from zero, the above equation written in polar coordinates will be :

$$\frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} (\sin \vartheta \, \mathfrak{a}_\varphi) \, r \frac{d\vartheta}{ds} + \frac{1}{r} \frac{\partial}{\partial r} (r \, \mathfrak{a}_\varphi) \frac{dr}{ds} = 0,$$

which may be transformed into

$$\frac{\partial}{\partial r} (r \sin \vartheta \, \mathfrak{a}_\varphi) \frac{dr}{ds} + \frac{\partial}{\partial \vartheta} (r \sin \vartheta \, \mathfrak{a}_\varphi) \frac{d\vartheta}{ds} = 0 \, ;$$

after integrating

$$\mathfrak{a}_\varphi r \sin \vartheta = \text{const.}$$

According to (8) the vector potential of the meridional currents is

$$\mathfrak{B}_\varphi = \beta \psi_2 P_2^{(1)},$$

which, written into the above equation and somewhat transformed, is

$$r \psi_2 (P_1 - P_3) = C.$$

The family of the curves appertaining to the different values of C is shown by fig. 2. We remark here that the selfsame family of curves produces also the trajectories of the meridional electric currents, mentioned in Chapter 4.3. These currents originate $\mathfrak{H}_\varphi$, the trajectories of which are the concentric circles running around the axis of rotation ; the equation of the circles in question is :

$$\left(\mathfrak{H}_\varphi, \frac{d\mathfrak{s}_\varphi}{ds} \right) = 0,$$

after integrating :

$$(\mathfrak{H}_\varphi, \mathfrak{s}_\varphi) = 0 \, ;$$

let us write it into polar coordinates :

The value of C varies from point to point in the meridional plane ; the curve joining the constant values of C represents the meridional sections of the force surfaces. This family of curves, as one may see it from the formula, agrees as to the form with the trajectories of the meridional currents.

Instead of the trajectories of the velocity field let us examine the family of surfaces of constant angular velocity. The equation of these may be easily obtained on the ground of (30a) :

$$\left(\frac{r_*}{r} \right)^2 \frac{\psi_2}{5{,}7608} - \frac{r^*}{r} \psi_3 \cos^2 \vartheta = C.$$

The family of curves is shown by fig. 3.

The equation of the lines of force of the magnetic field is obtained in a similar way as that of the meridional currents.

The equation of the lines of force of the internal magnetic field is :

$$(r^2 + 133{,}03 \, \psi_1 r) P_1^{(1)} P_1^{(1)} - 199{,}54 \psi_3 r P_1^{(1)} P_3^{(1)} = C,$$

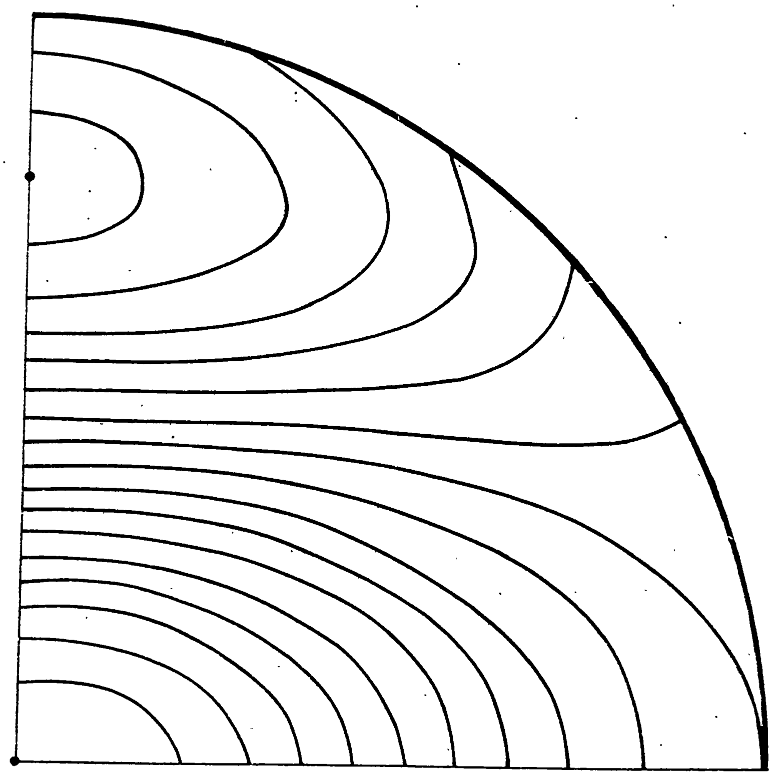

Fig. 3.

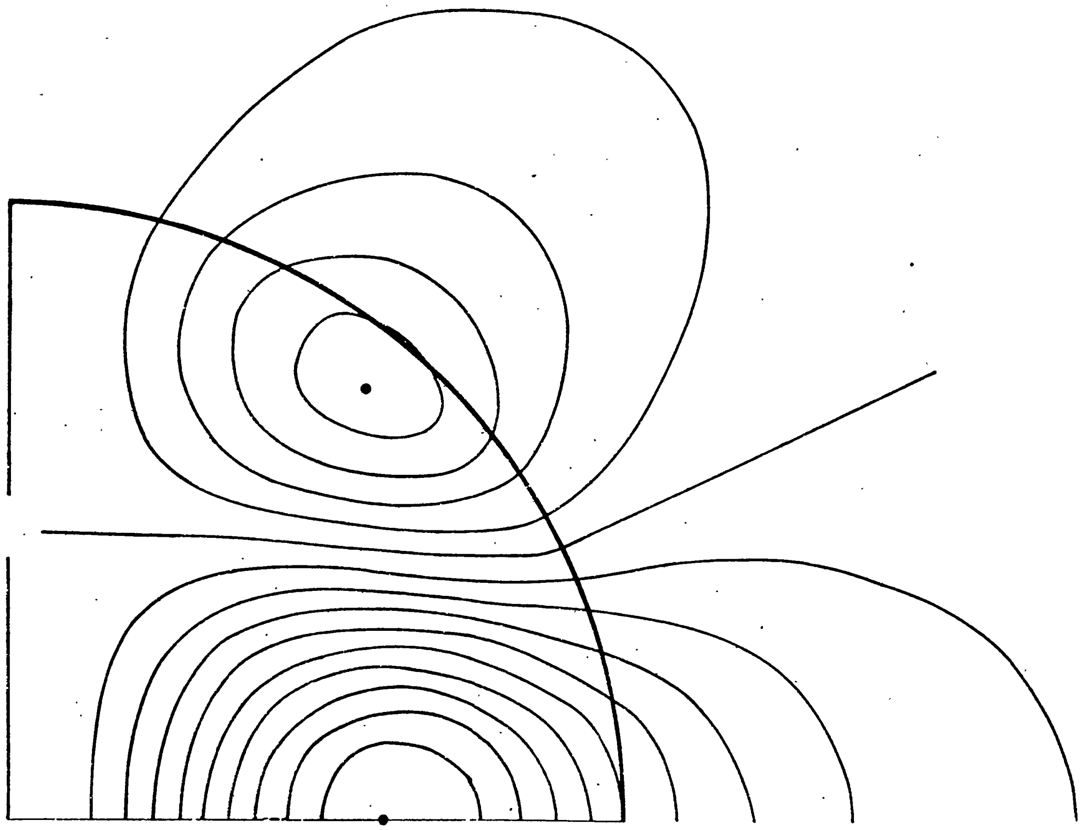

Fig. 4.

that of the external field (octopole-type field):

$$33{,}12 \frac{1}{r} P_1^{(1)} P_3^{(1)} = C.$$

the lines of force of the internal and external magnetic fields are shown by fig. 4.

Summary

We have determined the velocity distribution of a star and its magnetic field in first approximation. Our results are as follows.

Velocity distribution:

$$\mathfrak{v}_\varphi = \omega_0 r P_1^{(1)} + 0{,}968 \varkappa r_* (3 \psi_1 P_1^{(1)} - 2 \psi_3 P_3^{(1)}).$$

Distribution of the angular velocity:

$$\omega = \omega_0 + 83{,}66 \varkappa \frac{\psi_2}{r_2} - \frac{581{,}9 \varkappa}{r_*} \frac{\psi_3}{r} \cos^2 \vartheta.$$

Vector potential of meridional currents:

$$\mathfrak{B}_\varphi = 28{,}865 \varkappa \psi_2 P_2^{(1)}.$$

Vector potential of the internal magnetic field:

$$\mathfrak{A}_\varphi = H_0 r_* \left\{ \left[\frac{r}{r_*} + 133{,}03 \psi_1 \right] P_1^{(1)} - 119{,}54 \psi_3 P_3^{(1)} \right\}.$$

Vector potential of the external (octopole-type) field:

$$\mathfrak{A}_\varphi = -0{,}6631 H_0 \frac{r_*^5}{r^4} P_3^{(1)}.$$

Toroidal magnetic field strength:

$$\mathfrak{H}_\varphi = 59{,}74 H_0 \psi_2 P_2^{(1)}.$$

Appendix

1. The function ψ_l, used to solve the basic equations, is derivable from the Bessel functions:

$$\psi_l = r^{-1/2} J_{l+1/2} (\lambda r)$$

and therefore recurrence formulae, rules of differentiation and those of integration, relative to the Bessel functions, may be transformed also into ψ_l.

The important recurrence formula is

$$\psi_{l-1} + \psi_{l+1} = \frac{2l+1}{\lambda r} \psi_l .$$

Rules of differentiation :

$$\frac{d\psi_l}{dr} = \frac{\lambda l}{2l+1}\psi_{l-1} - \frac{\lambda(l+1)}{2l+1}\psi_{l+1},$$

$$\frac{1}{r}\frac{d}{dr}(r\psi_l) = \frac{\lambda(l+1)}{2l+1}\psi_{l-1} - \frac{\lambda l}{2l+1}\psi_{l+1}.$$

Integral formulae :

$$\int r^{l+2}\psi_l(\lambda r)\,dr = \frac{1}{2}r^{l+2}\psi_{l+1},$$

$$\int \frac{\psi_l\,dr}{r^{l-1}} = -\frac{\psi_{l-1}}{r^{l-1}}.$$

2. Tables for the functions ψ_l ($l = 0, 1, 2, 3, 4$). The functions ψ_l may be expressed by power functions and trigonometric ones :

$$\psi_0 = \frac{\sin x}{x},$$

$$\psi_1 = \frac{\sin x}{x^2} - \frac{\cos x}{x},$$

$$\psi_2 = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^3}\cos x,$$

$$\psi_3 = \left(\frac{15}{x^4} - \frac{6}{x^2}\right)\sin x - \left(\frac{15}{x^3} - \frac{1}{x}\right)\cos x,$$

$$\psi_4 = \left(\frac{105}{x^5} - \frac{45}{x^3} + \frac{1}{x}\right)\sin x - \left(\frac{105}{x^4} - \frac{10}{x^2}\right)\cos x.$$

In the neighbourhood of $x = 0$ we obtain the functional values by the aid of the subsequent series :

$$\psi_1 = \frac{x}{3}\left(1 - \frac{x^2}{10}\left(1 + \frac{x^2}{28}\left(1 - \frac{x^2}{54}\left(1 + \ldots\right)\right)\right)\right),$$

$$\psi_2 = \frac{x^2}{15}\left(1 - \frac{x^2}{14}\left(1 + \frac{x^2}{36}\left(1 - \frac{x^2}{66}\left(1 + \ldots\right)\right)\right)\right),$$

$$\psi_3 = \frac{x^3}{105}\left(1 - \frac{x^2}{18}\left(1 + \frac{x^2}{44}\left(1 - \ldots\right)\right)\right),$$

$$\psi_4 = \frac{x^4}{945}\left(1 - \frac{x^2}{22}\left(1 + \frac{x^2}{52}\left(1 - \ldots\right)\right)\right).$$

If $x > 2l + 1$, then for calculating ψ_l the following recurrence formula is expedient :

$$\psi_{l+1} = \frac{2l+1}{x}\psi_l - \psi_{l+1}.$$

In what follows we give the numerical values of ψ_0, ψ_1, ψ_2, ψ_3 and ψ_4, of which we have availed ourselves in the calculation.

x	ψ_0	ψ_1	ψ_2	ψ_3	ψ_4
0,0	1,000	0,000	0,000	0,000	0,000
0,2	0,933	0,066	0,003	0,000	0,000
0,4	0,974	0,131	0,011	0,001	0,000
0,6	0,941	0,193	0,023	0,002	0,000
0,8	0,897	0,250	0,041	0,005	0,000
1,0	0,841	0,301	0,062	0,009	0,001
1,2	0,777	0,345	0,087	0,015	0,002
1,4	0,704	0,381	0,113	0,023	0,004
1,6	0,625	0,409	0,142	0,034	0,006
1,8	0,541	0,427	0,170	0,064	0,009
2,0	0,455	0,435	0,198	0,061	0,014
2,2	0,366	0,435	0,225	0,080	0,020
2,4	0,282	0,425	0,249	0,095	0,027
2,6	0,198	0,406	0,270	0,113	0,035
2,8	0,120	0,379	0,287	0,133	0,045
3,0	0,047	0,346	0,299	0,152	0,056
3,2	— 0,018	0,306	0,306	0,171	0,069
3,4	— 0,075	0,262	0,307	0,188	0,081
3,6	— 0,123	0,215	0,302	0,205	0,096
3,8	— 0,161	0,166	0,292	0,218	0,110
4,0	— 0,189	0,116	0,276	0,229	0,125
4,2	— 0,208	0,067	0,256	0,237	0,139
4,4	— 0,216	0,021	0,230	0,241	0,153
4,6	— 0,216	— 0,023	0,201	0,241	0,166
4,8	— 0,208	— 0,062	0,169	0,238	0,177
5,0	— 0,192	— 0,095	0,135	0,230	0,187
5,2	— 0,170	— 0,123	0,099	0,218	0,194
5,4	— 0,143	— 0,144	0,063	0,202	0,199
5,6	— 0,113	— 0,159	0,028	0,183	0,201
5,8	— 0,080	— 0,167	— 0,006	0,161	0,201

Particular functional values :

the first zero of ψ_1 : $x_1 = 4{,}4936$

the first zero of ψ_2 : $x_2 = 5{,}7608$

corresponding values at the first zero of ψ_2 :

$$\psi_0(x_2) = -\ 0{,}0866 \qquad \psi_1(x_2) = -\ 0{,}166$$

$$\psi_3(x_2) = 0{,}166 \qquad \psi_4(x_2) = 0{,}201.$$

Budapest-Szabadsághegy, 15th June, 1955.